Polarized endomorphism of Fano varieties with complements

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BGS

Definition. Let X be a projective normal variety over $\mathbb{C}$. An surjective endomorphism $f\colon X \to X$ is a *polarized endomorphism* if there exists an ample divisor A on X such that $f^*A = mA$ for some integer $m > 1$.

Examples.

- $\mathbb{P}^n \xrightarrow{f_m} \mathbb{P}^n$; $[x_0 : \dots : x_n] \longmapsto [x_0^m : \dots : x_n^m]$

- Toric varieties. $f_m \rightsquigarrow m\mathbb{Z}^s \subseteq \mathbb{Z}^s$

- A abelian variety.

$$\begin{aligned}[m] : A &\longrightarrow A \\ a &\longmapsto ma\end{aligned}$$

Question.

What can we say about the geometry of a variety X if it admits a polarized endomorphism?

If X admits a polarized endomorphism, then

- $\kappa(X) \leq 0$. Meng '20

- if K_X is $\mathbb{Q}$-Cartier, then X has at worst lc singularities.

  Höring '14

- X satisfies Bott vanishing. Kawakami-Totaro '23.

  $H^i(X, \Omega^j(A)) = 0$ for $i>0$; $j>0$

Conjecture. Let X be a smooth Fano variety of Picard number 1. If X admits a non-isomorphic surjective endomorphism, then X is a projective space. '80

Conjecture. Let X be a variety of klt type. If X admits a polarized endomorphism, then X is a finite quotient of a toric fibration over an abelian variety.

Partial results.

- The conjecture is known for surfaces. Nakayama '02
- If X is a smooth Fano threefold admitting a polarized endomorphism, then X is toric. Sheng – Zhang – Zhong '22
- If X is a klt Calabi–Yau variety, then X is a finite quotient of an abelian variety. Meng '20
- If X is rationally connected smooth projective variety admitting a polarized endomorphism $f\colon X \to X$. Suppose there exists a reduced divisor $\Delta \subset X$ such that $f^{-1}(\Delta) = \Delta$ and $f|_{X\setminus\Delta}$ is étale. Then (X, Δ) is a toric pair. Meng - Zhang '19

Examples.

$$(\mathbb{P}^n, H_0 + \cdots + H_n) \xrightarrow{f_m} (\mathbb{P}^n, H_0 + \cdots + H_n)$$

$$f_m^*(K_{\mathbb{P}^n} + \sum H_i) = K_{\mathbb{P}^n} + \sum H_i$$

Take $G \leq S_{n+1}$ with $n = p^q$, p prime.

(Kollár–Xu, Saltman) $\rightarrow$ f_m descends to $(\mathbb{P}^n/G, \sum H_i/G)$

but $\mathbb{P}^n/G$ is not rational

Defn: $f : (X, \Delta) \longrightarrow (X, \Delta)$ is polarized if $f : X \to X$ is polarized and $f^*(K_X + \Delta) = K_X + \Delta$.

Theorem.

big & nef

Let X be a Fano type variety. $-(K_X+B)$ ample for some $B \geq 0$

Let (X, Δ) be a log Calabi–Yau variety with $K_X + \Delta \sim 0$.

If (X, Δ) admits a polarized endomorphism, then (X, Δ) is a finite quotient of a log Calabi–Yau toric pair.

Step 1. Lifting polarized endomorphisms to finite covers.

$$\begin{tikzcd} (Y, \Delta_Y) \arrow[d, "g"] \\ (X, \Delta) \end{tikzcd} \quad \text{finite cover s.t.} \quad \begin{tikzcd} U_Y \arrow[d, "\tilde{g}"] \\ U := X^{reg} \setminus \Delta \end{tikzcd} \quad \text{étale}$$

$$\begin{tikzcd} \cdots \arrow[r] & U_{Y_2} \arrow[r] \arrow[d] & U_{Y_1} \arrow[r] \arrow[d] & U_Y \arrow[d, "\tilde{g}"] \\ \cdots \arrow[r] & U \arrow[r, "f|_U"] & U \arrow[r, "f|_U"] & U \end{tikzcd}$$

$$\begin{tikzcd} U_{Y_n} \arrow[r] \arrow[d, "\tilde{g}_n"'] & U_Y \arrow[d, "\tilde{g}"] \\ U \arrow[r, "f^n|_U"] & U \end{tikzcd} \qquad \deg(\tilde{g}_i) = \deg(\tilde{g})$$

Step 1. Lifting polarized endomorphisms to finite covers.

Each $\tilde{g}_i \rightsquigarrow H_i \leq \pi_1(U)$ of index $\deg(\tilde{g})$.

- $\pi_1(U)$ is finitely presented $\Rightarrow$ there are finitely many subgroups of index $\deg(\tilde{g})$.

- For $n \gg 0$ $\quad U_{Y_n} \cong U_Y$

$$\begin{tikzcd}
(Y_n, \Delta_{Y_n}) \arrow[r] \arrow[d] & (Y, \Delta_Y) \arrow[d, "g"] \\
(X, \Delta) \arrow[r, "f^n"'] & (X, \Delta)
\end{tikzcd}$$

$(Y, \Delta_{Y_n}) \cong (Y, \Delta_Y)$

$\Rightarrow (Y, \Delta_Y) \xrightarrow{f_Y} (Y, \Delta_Y)$

Step 2. Results on the étale fundamental group of $U = X^{\text{reg}} \setminus \Delta$.

Want to show: $\pi_1^{alg}(U)$ is virtually abelian

there exists $A \leq \pi_1^{alg}(U)$ normal abelian of finite index.

Let (X, D) s.t. $D = \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) D_i$.

$\pi_1(X, D)$ " $:=$ " $\pi_1(X^{reg} \setminus \text{supp}(D)) / \langle \gamma_1^{m_1}, \gamma_2^{m_2}, \ldots, \gamma_s^{m_s} \rangle_{nor}$

(Braun '21, Braun-Filipazzi-Moraga-Svaldi '22)

$\pi_1(X, D)$ are finite and they have a normal abelian subgroup of bounded index.

Step 2. Results on the étale fundamental group of $U = X^{\mathrm{reg}} \setminus \Delta$.

$$\pi_1^{alg}(U) = \varprojlim_{\mathbb{Z}_{\geq 1}^s \ni \vec{m}} \pi_1(X, \Delta_{\vec{m}}) \quad ; \quad \Delta_{\vec{m}} = \sum \left(1 - \frac{1}{m_i}\right) \Delta_i$$

$\Rightarrow$ There exists $A \trianglelefteq \pi_1^{alg}(U)$ abelian of finite index.

Step 3. Study Galois polarized endomorphisms.

For A, take $U' \to U$ induced cover.

$$
\begin{tikzcd}
(Y, \Delta_Y) \arrow[r, "f_Y"] \arrow[d] & (Y, \Delta_Y) \arrow[d] \\
(X, \Delta) \arrow[r, "f^n"'] & (X, \Delta)
\end{tikzcd}
$$

- Now $\pi_1^{alg}(Y^{reg} \setminus \Delta_Y)$ is abelian.

$\Rightarrow$ f_Y is Galois

$\rightsquigarrow$ $G \leq \mathrm{Aut}(Y, \Delta_Y)$

$$
\begin{tikzcd}
(Y, \Delta_Y) \arrow[r, "/G"] \arrow[dr, "f_Y"'] & (Y/G, \Delta_Y/G) \arrow[d, "\cong"] \\
& (Y, \Delta_Y)
\end{tikzcd}
$$

Step 3. Study Galois polarized endomorphisms.

Because Y is also of Fano type

$$1 \longrightarrow \underbrace{\mathbb{T}}_{\text{alg torus}} \longrightarrow \operatorname{Aut}(Y, \Delta_Y) \longrightarrow \underbrace{F}_{\text{Finite}} \longrightarrow 1$$

- For some $n \gg 0$, $f_Y^n \rightsquigarrow G_n \leq \mathbb{T}$

$$N_{\operatorname{Aut}}(G_n)/G_n \longrightarrow \operatorname{Aut}(Y/G_n, \Delta_Y/G_n)$$

$$\operatorname{Aut}(Y/G_n, \Delta_Y/G_n) \cong \operatorname{Aut}(Y, \Delta_Y)$$

Step 4. Showing that (Y, Δ_Y) is a toric pair.

$G \leq \mathbb{T} \leq \mathrm{Aut}(Y, \Delta_Y)$

We want to show: $\mathrm{rank}\, \mathbb{T} = \dim Y$.

Assume not, then $\mathrm{rank}\, \mathbb{T} < \dim Y$. Then there exists a subvar of $\mathrm{codim} \geq 2$ which is fixed by $\mathbb{T}$.

As you take $/G_m$ for $m >> 0$ you create worse and worse sing.

Step 4. Showing that (Y, Δ_Y) is a toric pair.

But $(Y/\mathbb{G}_m, \Delta_Y/\mathbb{G}_m) \cong (Y, \Delta_Y)$

$\vdots$

$\Rightarrow (Y, \Delta_Y)$ is a toric pair.

Thank you!